Mathematical groundwork I: Fourier theory

Fundamentals of Radio Interferometry

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NASSP 2016

Basic remarks

- Mathematics in this course is a requirement to understand and conduct interferometric imaging
- Interferometry is a nice field for the mathematically inclined, but required maths is manageable
- Mathematics is presented as tool, proofs partly not complete and used as an exercise to memorize the tool functions
- Principles presented here are fundamental to experimental physics, radio technology, informatics, image processing, theoretical physics etc.
- Unlike the last session, this one will not contain many pictures

Important functions

- Some functions that will return over and over again are presented

Important functions: Gaussian

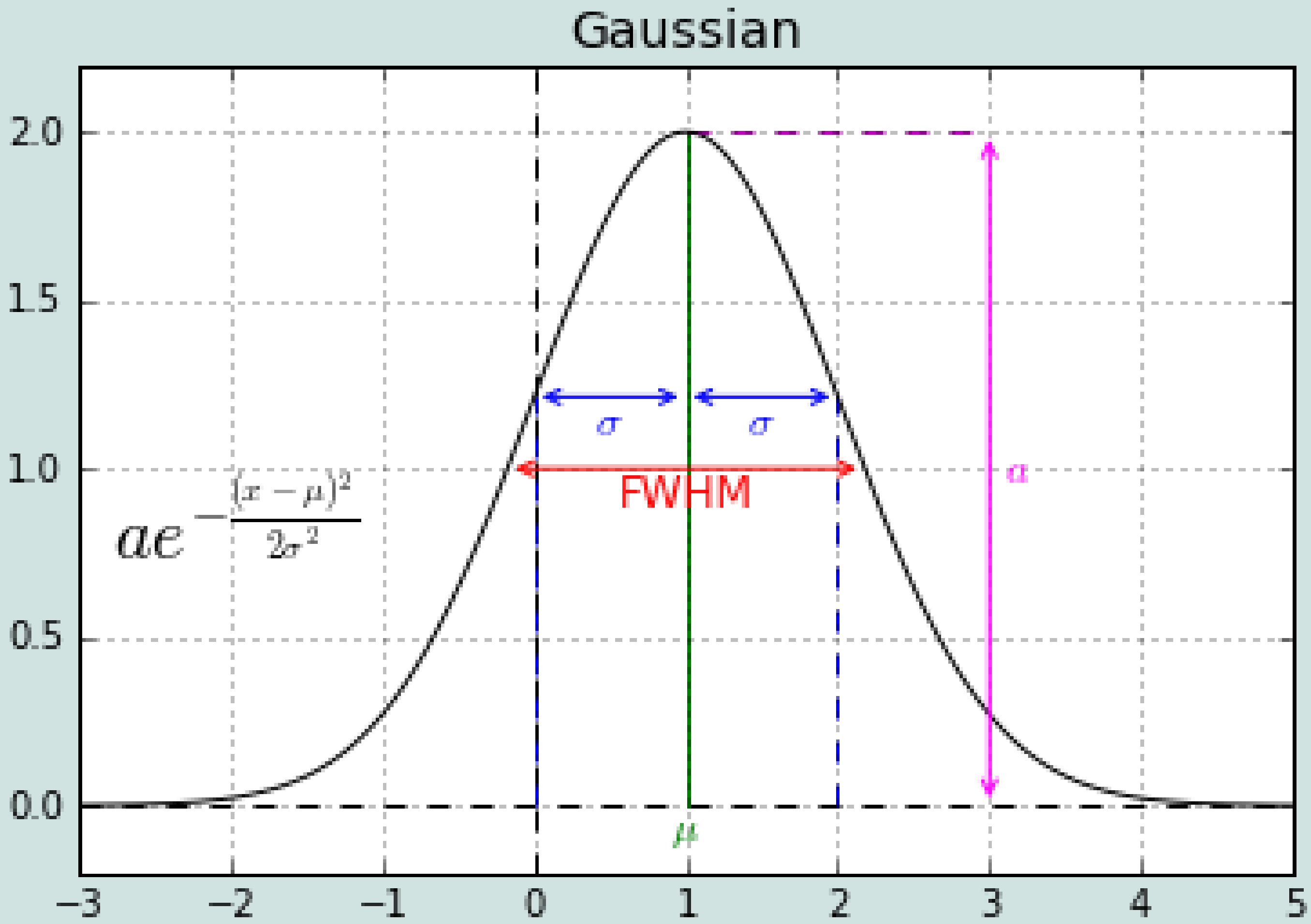
$$f(x) = ae^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

a: amplitude

μ: mean

σ: standard deviation

Carl- Friedrich Gauß
(1777- 1855)



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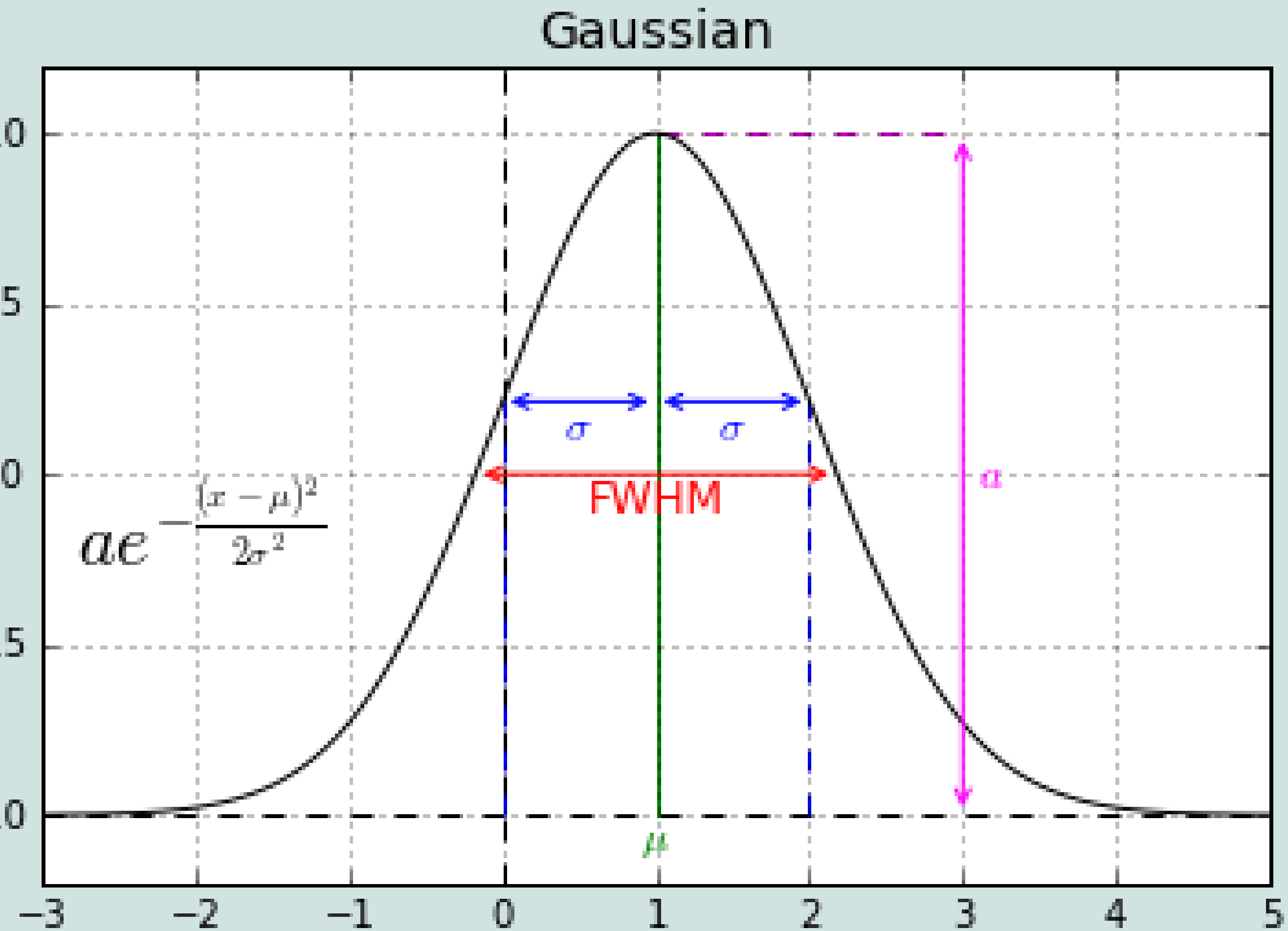

FWHM (full width at half maximum):

$$ae^{-\frac{x^2}{2\,\sigma^2}} = \frac{a}{2} \Leftrightarrow$$

$$x_{1,2} = \pm\sqrt{2\ln(2)}\,\sigma \Rightarrow$$

$$FWHM = 2\sqrt{2\ln(2)}\,\sigma \approx 2.35$$

Important functions: Gaussian

$$f(x) = ae^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

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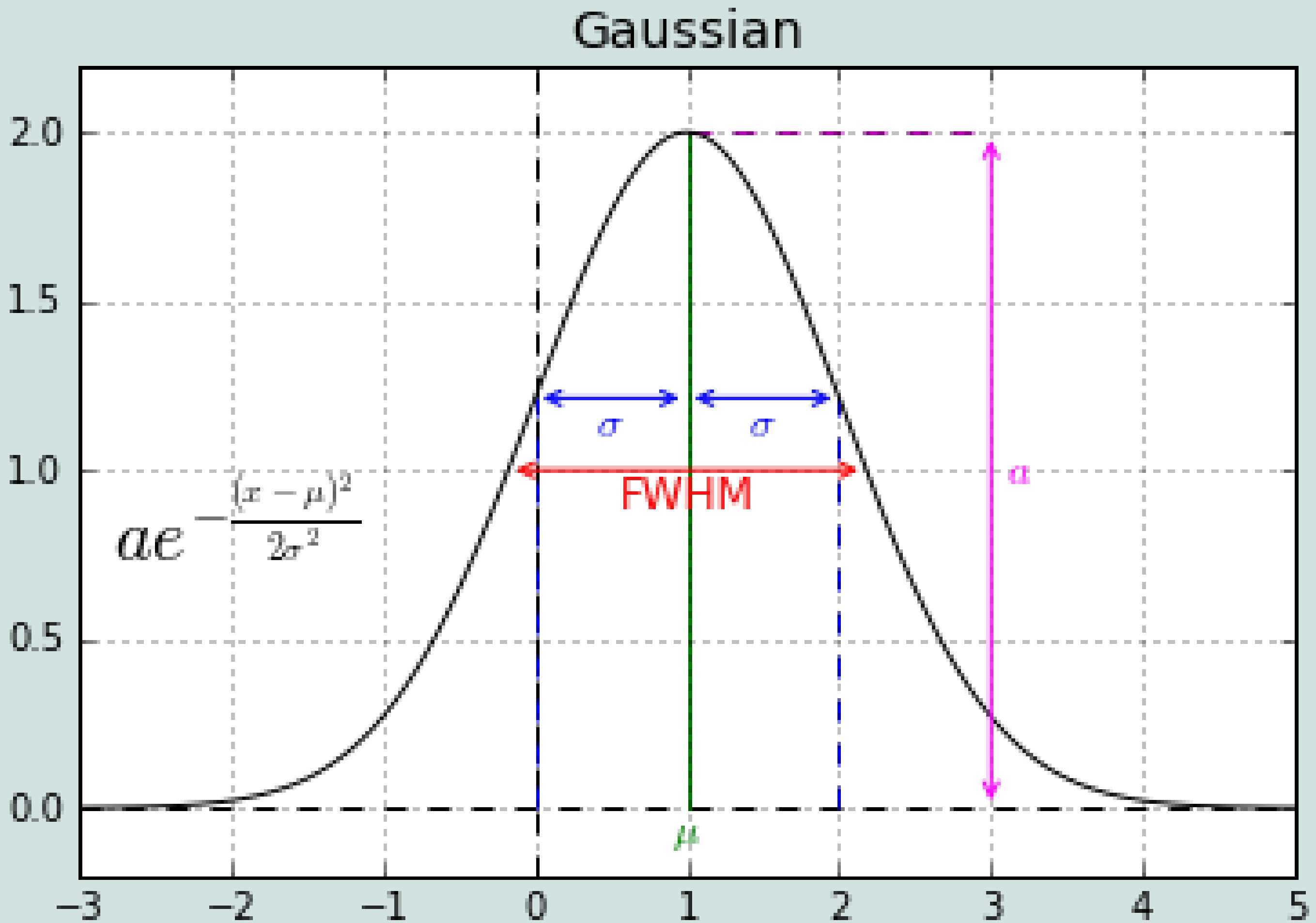
Area below Gaussian:

$$\int_{-\infty}^{+\infty} ae^{-\frac{x^2}{2\,\sigma^2}}\,dx = a\sqrt{2\pi}\sigma$$

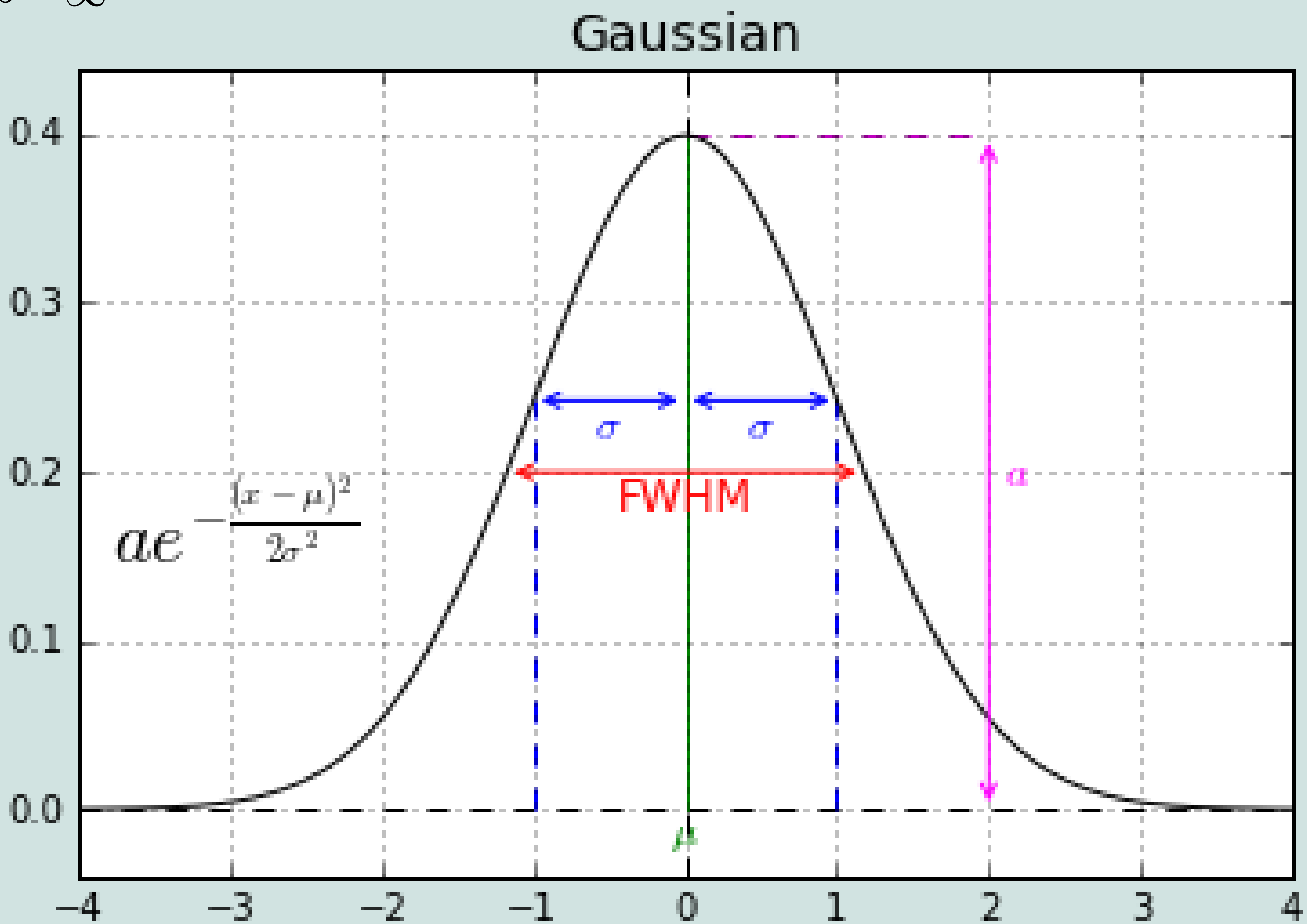


Important functions: Normalised Gaussian

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \Rightarrow \int_{-\infty}^{+\infty} f(x)dx = 1$$

μ: mean

σ: standard deviation



Important functions: Gaussian 2d

$$f(\mathbf{x}) = f(x_1, x_2) = e^{-\left(\frac{x_1^2}{2\,\sigma_1^2} + \frac{x_2^2}{2\,\sigma_2^2}\right)}$$

$$\begin{aligned}
\mathbf{A} &= (A_{ij}) \\
&= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \\
&= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\left(\mathbf{A}^T\right)_{ij} &= (A)_{ji} \\
&= \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} \\
&= \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
(h(\mathbf{A})(\mathbf{x}))_i &= (h(\mathbf{A})(x_1, x_2))_i \\
&= (\mathbf{A} \cdot \mathbf{x})_i \\
&= \sum_{j=1}^{2} A_{ij} x_j
\end{aligned}$$

$$\begin{aligned}
g(\mathbf{x}) &= f(h^{-1}(\mathbf{A})(\mathbf{x})) \\
&= f(h(\mathbf{A}^{-1})(\mathbf{x})) \\
&= f(h(\mathbf{A}^T)(\mathbf{x}))
\end{aligned}$$

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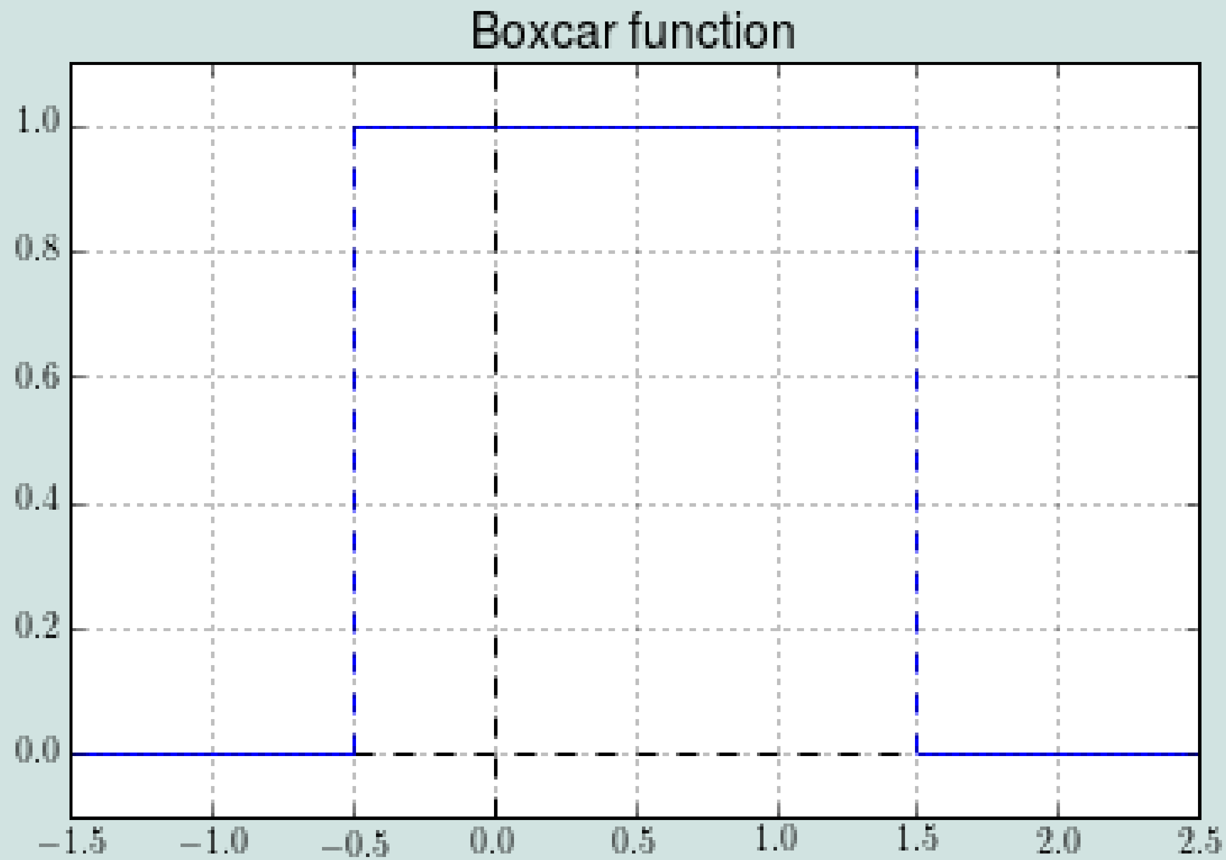
&= f(h(\mathbf{A}^{T})(\mathbf{x}))
\end{aligned}$$

$$\begin{aligned}
\int g(\mathbf{x})\, d^2\mathbf{x} &= \int f(h^{-1}(\mathbf{A})(\mathbf{x}))\, d^2\mathbf{x} \\
&= 2\pi\sigma_1\sigma_2 \\
&= 2\pi \frac{FWHM_1}{\sqrt{\ln 256}} \frac{FWHM_2}{\sqrt{\ln 256}} \\
&\approx 1.13309\, FWHM_1\, FWHM_2
\end{aligned}$$

Important functions: Boxcar function

$$\sqcap_{a,b}(x) = \begin{cases} 0 & \text{for} \quad x < a \\ 1 & \text{for} \quad a \leq x \leq b \\ 0 & \text{for} \quad x > b \end{cases}$$

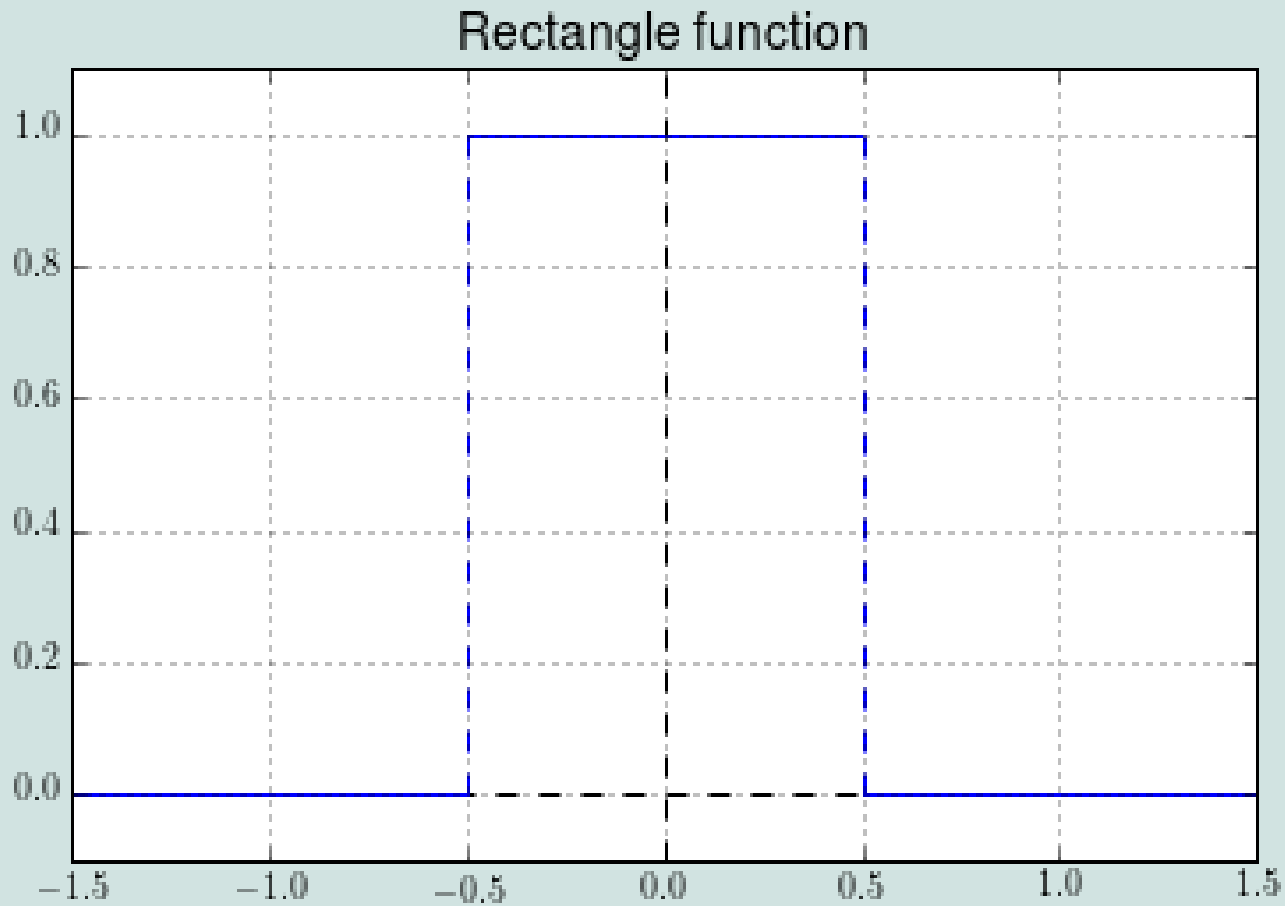
$$\Pi_{a,\,b}(x) = \begin{cases} 0 & \text{for} \quad x < a \\ 1 & \text{for} \quad a \leq x \leq b \\ 0 & \text{for} \quad x > b \end{cases}$$



Important functions: Rectangle function

$$\sqcap(x) = \sqcap_{-\frac{1}{2},\frac{1}{2}}$$
$$= \begin{cases} 0 & \text{for} \quad x < -\frac{1}{2} \\ 1 & \text{for} \quad -\frac{1}{2} \leq x \leq \frac{1}{2} \\ 0 & \text{for} \quad x > \frac{1}{2} \end{cases}$$

$$\Pi(x) = \begin{cases} 0 & \text{for} \quad x < -\frac{1}{2} \\ 1 & \text{for} \quad -\frac{1}{2} \leq x \leq \frac{1}{2} \\ 0 & \text{for} \quad x > \frac{1}{2} \end{cases}$$



Important functions: Sinc

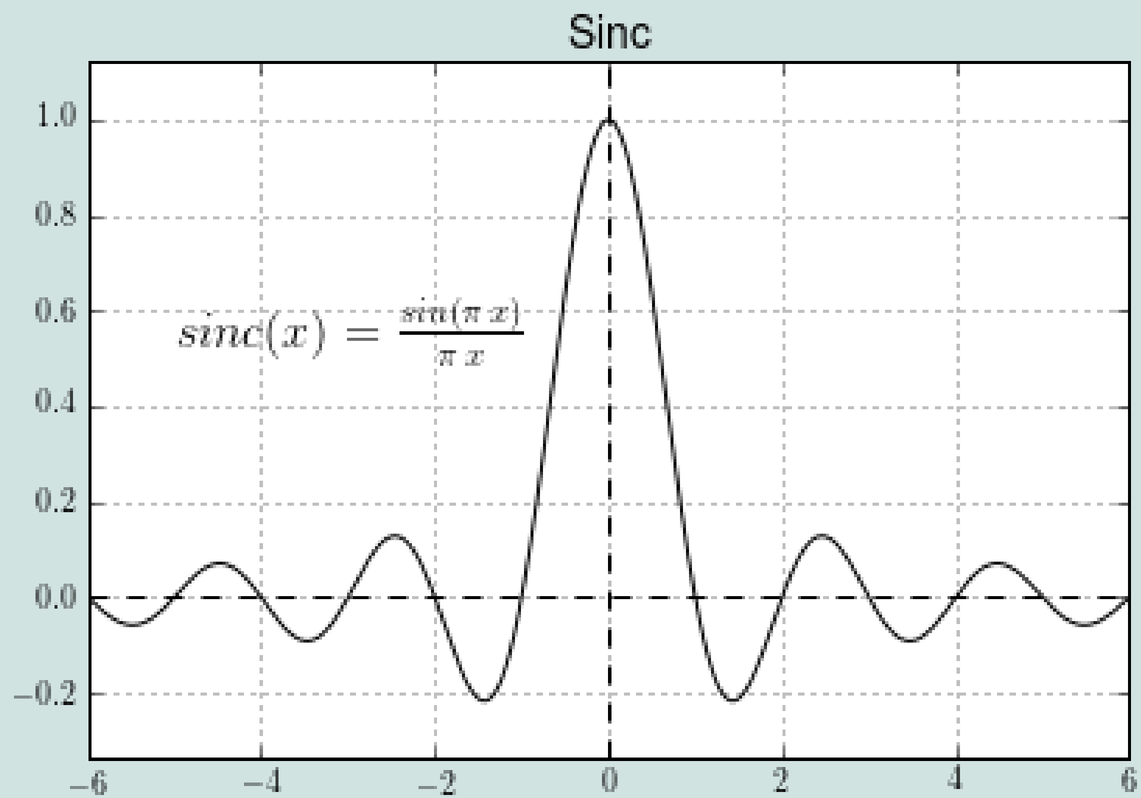
$$sinc_{\mathrm{u}}(x) = \begin{cases} 1 & \text{for} \quad x = 0 \\ \frac{sin x}{x} & \text{for} \quad x \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} sinc_{\mathrm{u}}(x)\, dx = \pi$$

$$sinc(x) = sinc_{\mathrm{u}}(\pi x)$$

$$= \begin{cases} 1 & \text{for} \quad x = 0 \\ \frac{sin(\pi x)}{\pi x} & \text{for} \quad x \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} sinc(x)\, dx = 1$$

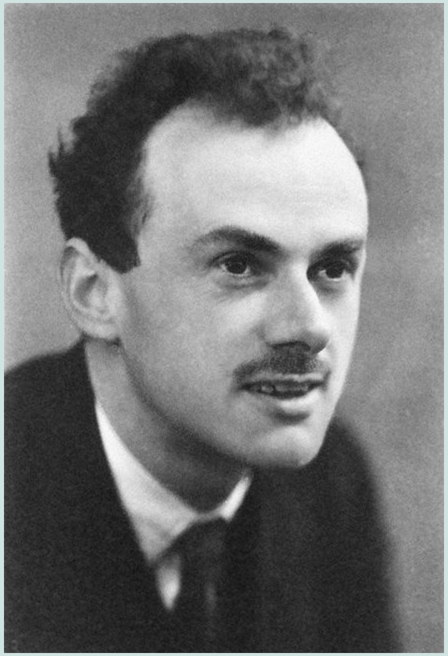


Important functions: Dirac's Delta

- Not a function but a distribution
- In many ways a function...

$$\forall x \in \mathbb{R},\, x \neq 0 \Rightarrow \delta(x) = 0$$

$$\forall f \in \mathbb{C}^0,\, \varepsilon \in \mathbb{R},\, \varepsilon > 0 \quad \int_{-\varepsilon}^{+\varepsilon} f(x)\delta(x)\,dx = f(0)$$



Paul Dirac
(1902- 1984)

It follows:

$$\varepsilon \in \mathbb{R},\, \varepsilon > 0 \;\Rightarrow \int_{-\varepsilon}^{+\varepsilon} \delta(x)\,dx$$

$$= \int_{-\varepsilon}^{+\varepsilon} \mathbf{1} \cdot \delta(x)\,dx$$

$$= \mathbf{1}(0)$$

$$= 1$$

Important functions: Dirac's Delta

- More precisely, the Delta function is defined as the “limit” of a suitable series

Find set of functions with δ_α with

$$\lim_{a\to 0}\int_{-\infty}^{-\varepsilon} f(x)\delta_a(x)\,dx = 0$$

$$\lim_{a\to 0}\int_{-\varepsilon}^{+\varepsilon} f(x)\delta_a(x)\,dx = f(0)$$

$$\lim_{a\to 0}\int_{+\varepsilon}^{+\infty} f(x)\delta_a(x)\,dx = 0$$

Then define δ through the integral

$$\int_{-\varepsilon}^{+\varepsilon} f(x)\delta(x)\,dx = \lim_{a\to 0}\int_{-\varepsilon}^{+\varepsilon} f(x)\delta_a(x)\,dx$$

In this sense:

$$\delta(x) = \lim_{a\to 0}\delta_a(x)$$

Important functions: Dirac's Delta

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$$\lim_{a\to 0}\int_{+\varepsilon}^{+\infty} f(x)\delta_a(x)\,dx = 0$$

$$\delta(x) = \lim_{a\to 0}\frac{1}{\sqrt{2\pi a}}e^{-\frac{(x-\mu)^2}{2a^2}} = \lim_{a\to 0}\frac{1}{\pi}\frac{a}{x^2+a^2} = \lim_{a\to 0}\frac{1}{a}sinc(\frac{x}{a})$$

Then define δ through the integral

$$\int_{-\varepsilon}^{+\varepsilon} f(x)\delta(x)\,dx = \lim_{a\to 0}\int_{-\varepsilon}^{+\varepsilon} f(x)\delta_a(x)\,dx$$

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Important functions: Dirac's Delta

- More precisely, the Delta function is defined as the “limit” of a suitable series

$$\delta(x) = \lim_{a\to 0} \frac{1}{\sqrt{2\pi}a} e^{-\frac{(x-\mu)^2}{2a^2}}$$
$$= \lim_{a\to 0} \frac{1}{\pi} \frac{a}{x^2 + a^2}$$
$$= \lim_{a\to 0} \frac{1}{a} sinc(\frac{x}{a})$$

- Two important relations:

$$\int_{a-\varepsilon}^{a+\varepsilon} f(x)\delta(x-a)\,dx \;=\; f(a)$$

$$\delta(ax) \;=\; \frac{1}{|a|}\delta(x)$$

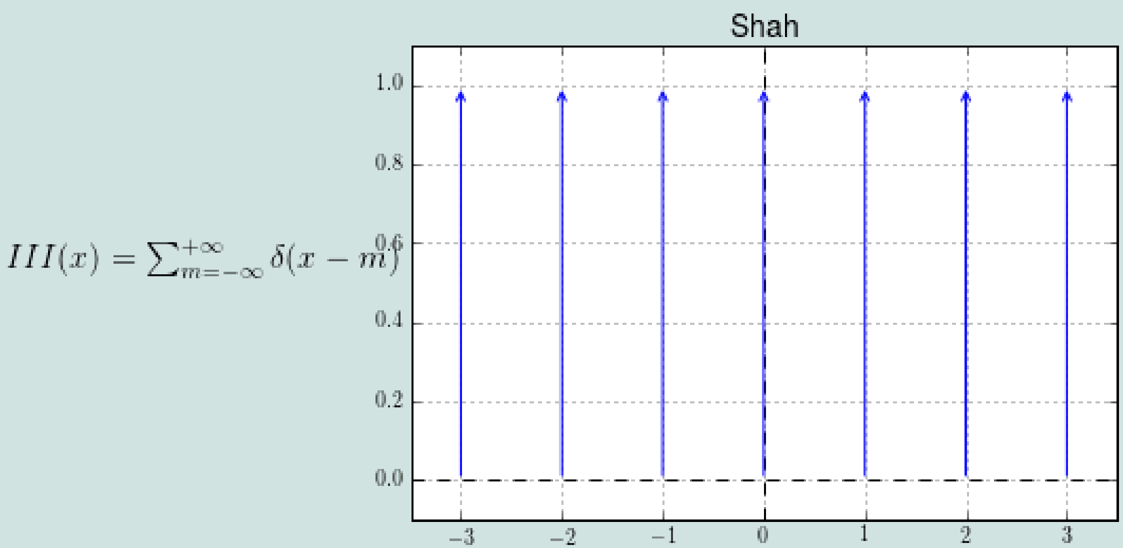
Important functions: Sha (Shah) or comb function

$$III(x) = \sum_{m=-\infty}^{+\infty} \delta(x - m)$$

Relations:

$$\begin{aligned} III_a(x) &= III(ax) \\ &= \sum_{m=-\infty}^{+\infty} \delta(ax - m) \\ &= \sum_{m=-\infty}^{+\infty} \delta\left(a\left(x - \frac{m}{a}\right)\right) \\ &= \sum_{m=-\infty}^{+\infty} \frac{1}{|a|}\delta\left(x - \frac{m}{a}\right) \end{aligned}$$

$$\begin{aligned} III_a(-x) &= III_a(x) \\ III_a\left(x + \frac{n}{a}\right) &= III_a(x) \end{aligned}$$



The Fourier transform

- Definition of the Fourier transform:

$$\mathscr{F} : [\mathbb{R} \to \mathbb{C}] \to [\mathbb{R} \to \mathbb{C}]$$

$$\forall f : \mathbb{R} \to \mathbb{C}, \int_{-\infty}^{+\infty} |f(x)| \, dx \in \mathbb{R}$$

$$\mathscr{F}\{f\}(s) = \int_{-\infty}^{+\infty} f(x) \, e^{-\imath 2\pi x s} dx$$

- Inverse Fourier transform (via the Fourier inversion theorem):

$$\forall F : \mathbb{R} \to \mathbb{C}, \int_{-\infty}^{+\infty} |F(s)| \, ds \in \mathbb{R}$$

$$\mathscr{F}^{-1}\{F\}(x) = \int_{-\infty}^{+\infty} F(s) \, e^{\imath 2\pi x s} ds$$

$$\Rightarrow \mathscr{F}^{-1}\{\mathscr{F} f\}(x) = f(x) \wedge \mathscr{F}\{\mathscr{F}^{-1} F\}(s) = F(s)$$

Jean-Baptiste Joseph Fourier

(1768 - 1830)

The Fourier transform

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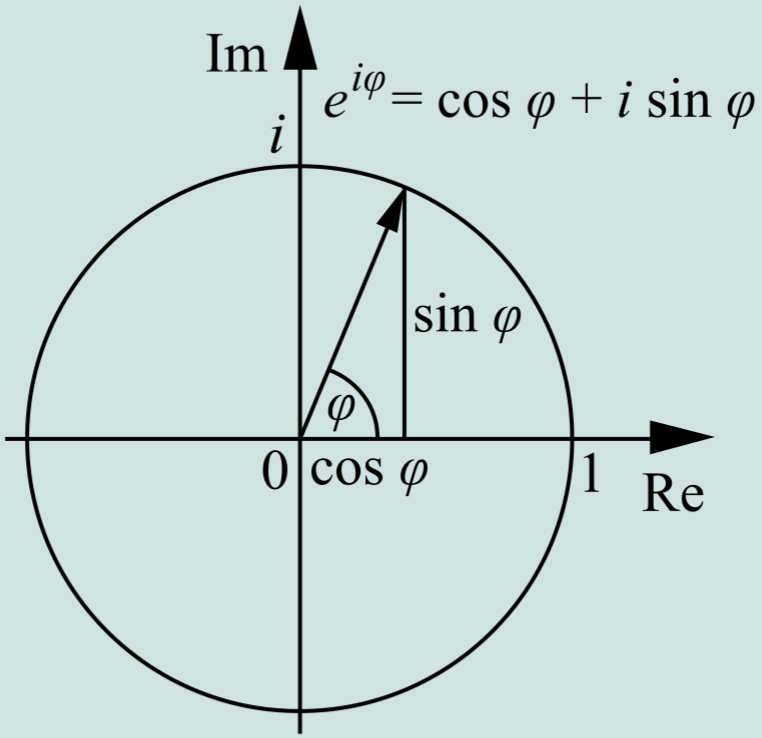
- The Fourier tansform can be seen as decomposition of a function into a wave package



Leonard Euler
(1707 - 1783)

The Fourier transform

- Notation: functions in the "Fourier space" are named by capital letters

$$\mathscr{F}\{f\}(s) = F(s) \Rightarrow f \rightleftharpoons F$$

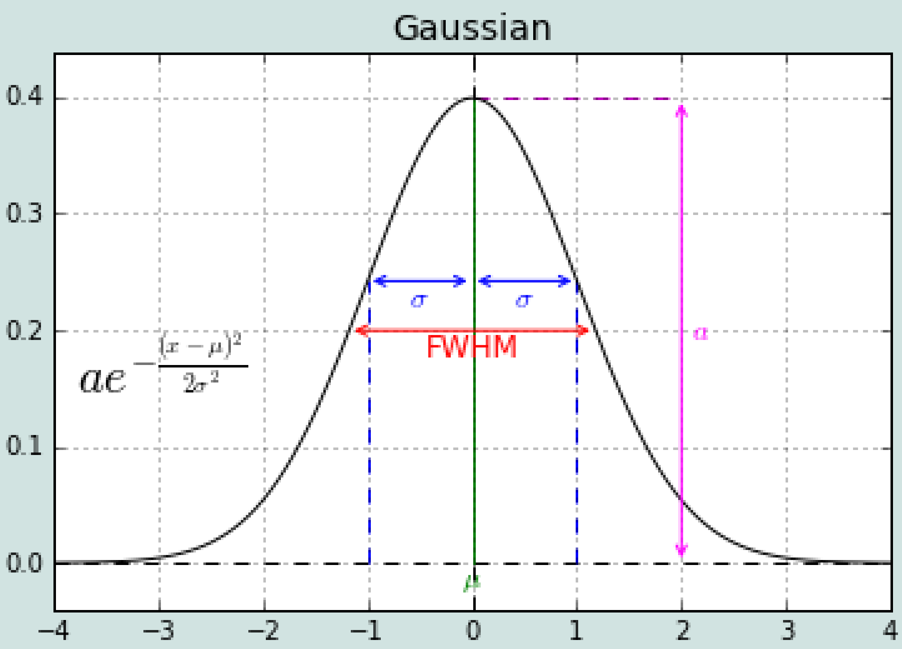
- The inverse Fourier transform is the Fourier transform of the reverse function (an inverse Fourier transform is hence a triple forward Fourier transform)

$$f_-(x) = f(-x)$$

$$\begin{aligned}
\mathscr{F}\{f\}(s) &= \int_{-\infty}^{+\infty} f(x) e^{-\imath x s} dx \\
&\underset{x'=-x}{=} \int_{+\infty}^{-\infty} f(-x') e^{\imath x' s} \frac{dx}{dx'}\, dx' \\
&= \int_{+\infty}^{-\infty} f(-x') e^{\imath x' s}\, dx' \\
&= \mathscr{F}^{-1}\{f_-\}(s)
\end{aligned}$$

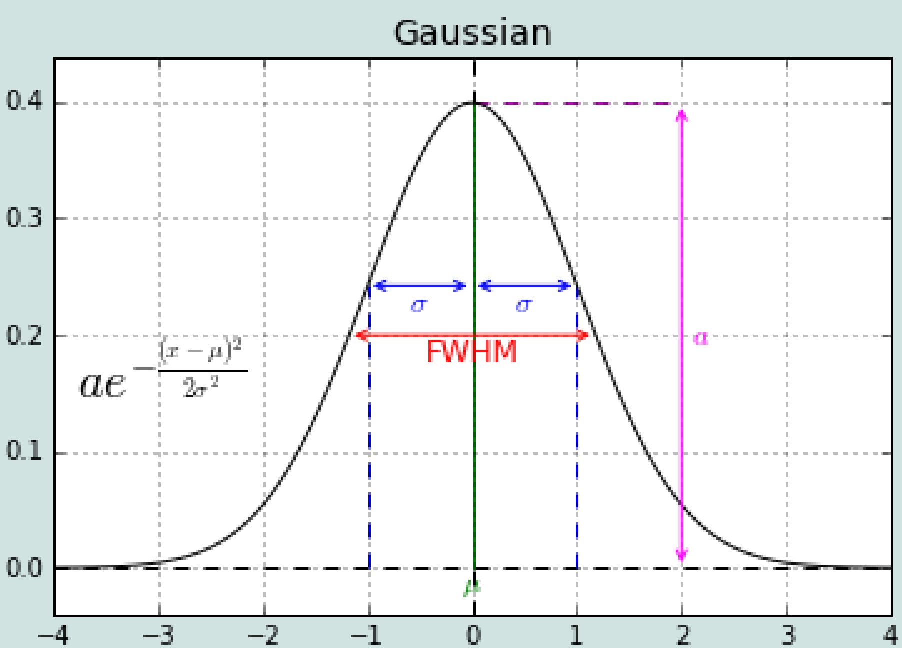
The Fourier transform of a Gaussian

- The Fourier transform of a Gaussian with dispersion σ_x is ...

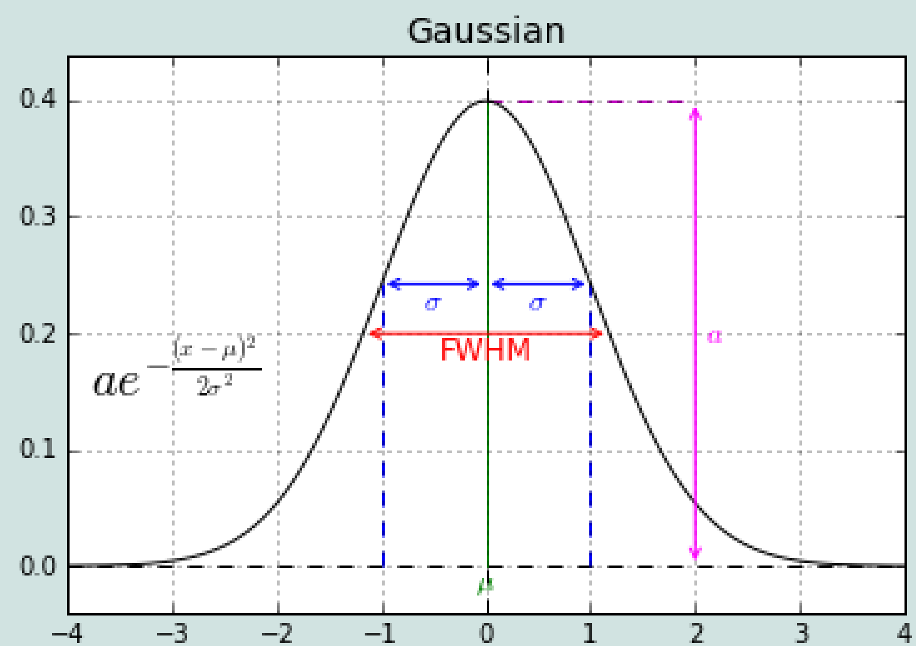


The Fourier transform of a Gaussian

- The Fourier transform of a Gaussian with dispersion σ_x is a Gaussian with dispersion $\sigma_s=(2\pi\sigma_x)^{-1}$



$\rightleftharpoons$



The Fourier transform of a Delta function

- The Fourier transform of a Delta function is ...

The Fourier transform of a Delta function

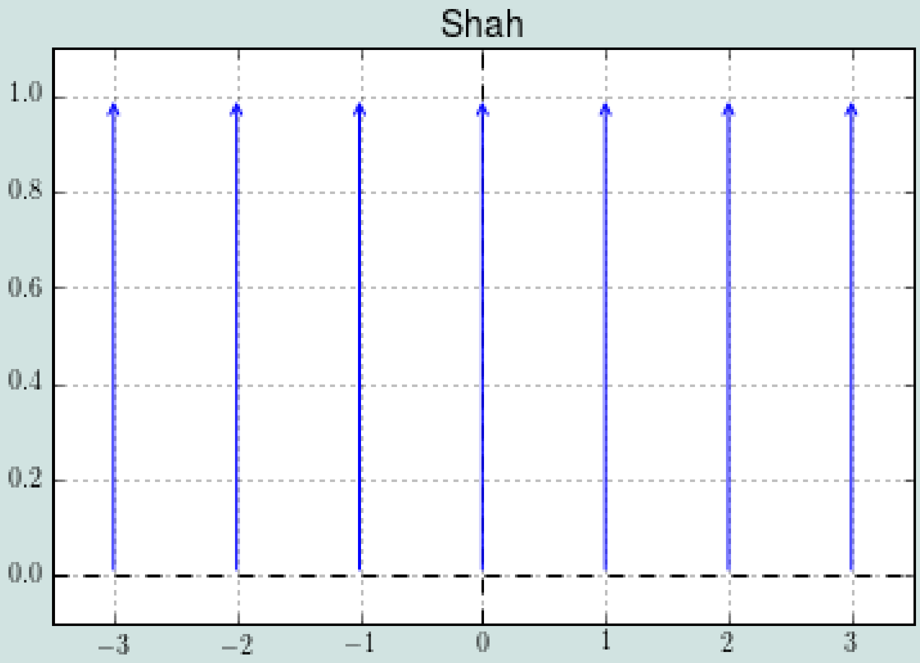
- The Fourier transform of a Delta function is a sinusoid in the real and the imaginary part, a wave

$$
\begin{aligned}
\mathscr{F}\{\delta\}(s) &= \int_{-\infty}^{+\infty} \delta(x)\, e^{-\imath 2\pi x s} dx \\
&= e^{0} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{F}\{\delta_{x_0}\}(s) &= \int_{-\infty}^{+\infty} \delta(x - x_0)\, e^{-\imath 2\pi x s} dx \\
&= e^{-\imath 2\pi x_0 s} \\
&= \cos\left(2\pi x_0 s\right) - \imath \sin\left(2\pi x_0 s\right)
\end{aligned}
$$

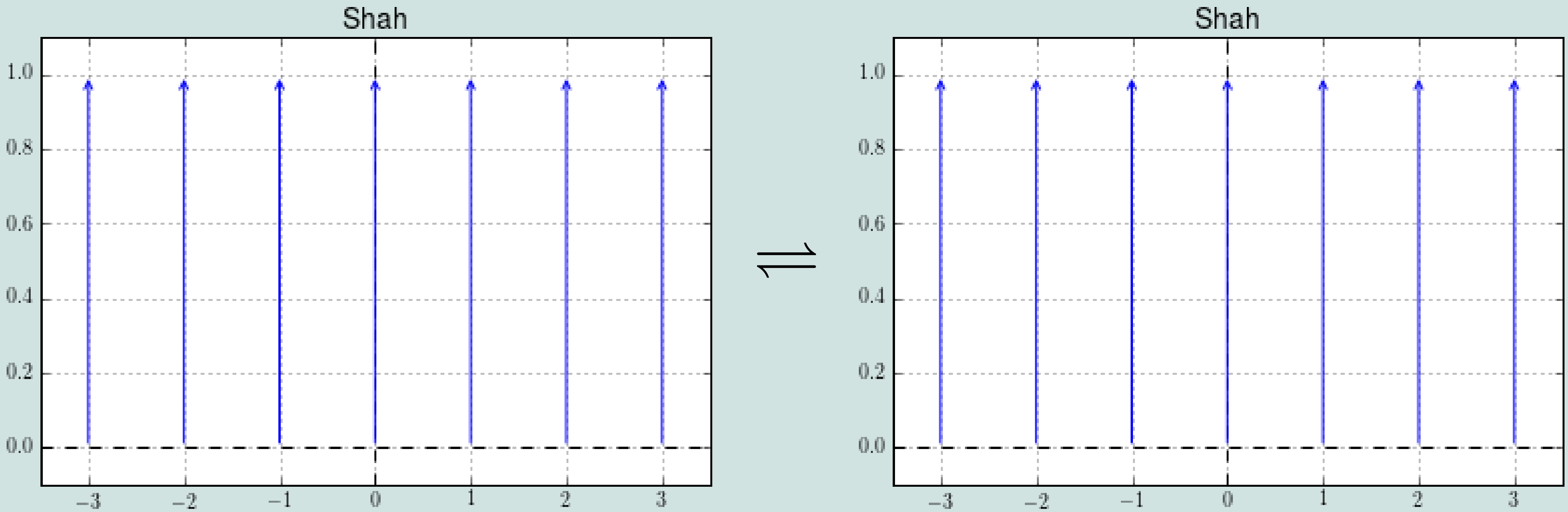
The Fourier transform of a comb function

- The Fourier transform of a sha function with period T is ...



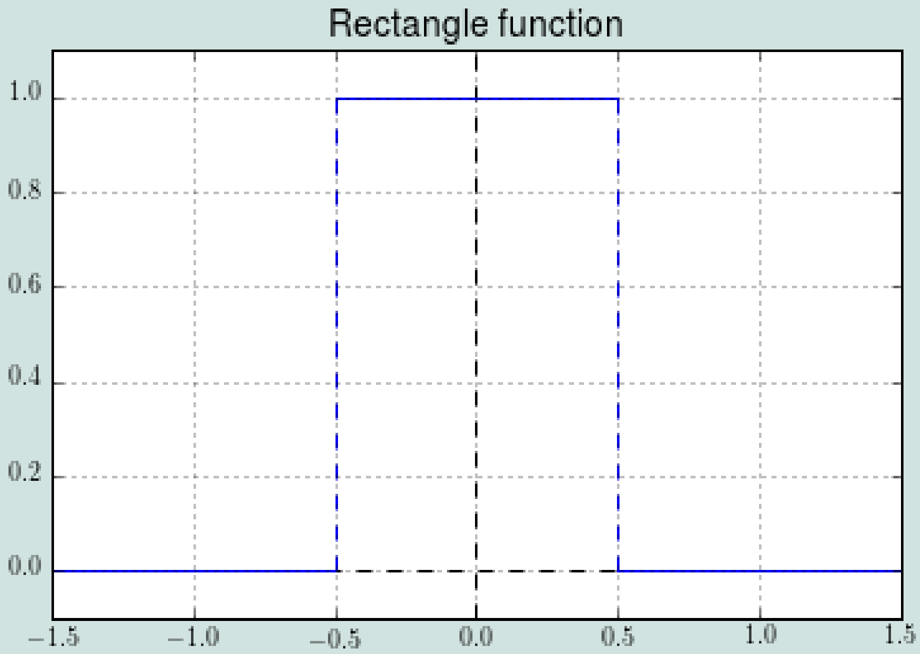
The Fourier transform of a comb function

- The Fourier transform of a sha function with period T is a sha function with period T^{-1}



The Fourier transform of a rectangle function

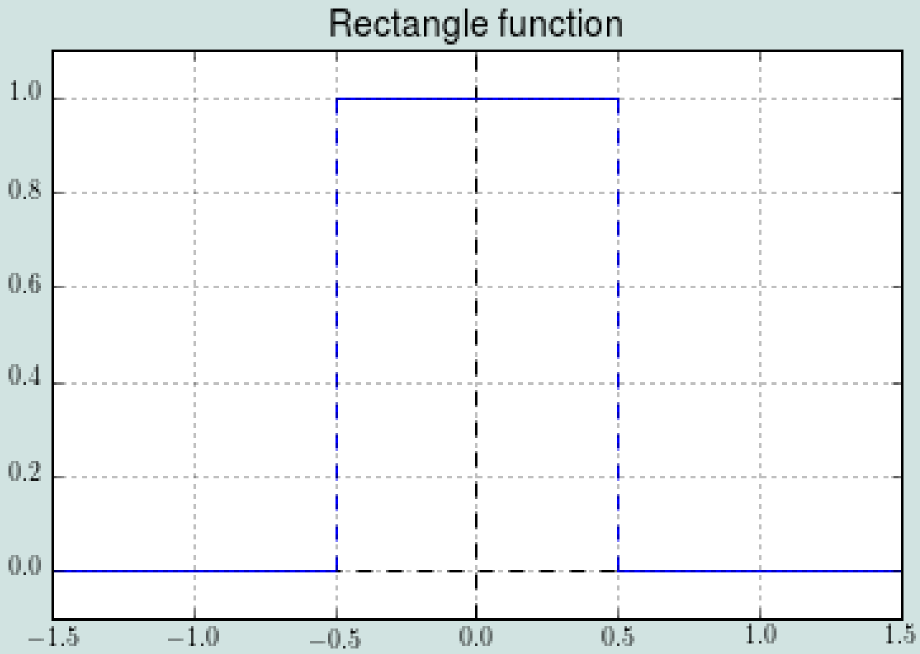
- The Fourier transform of a rectangle function is ...



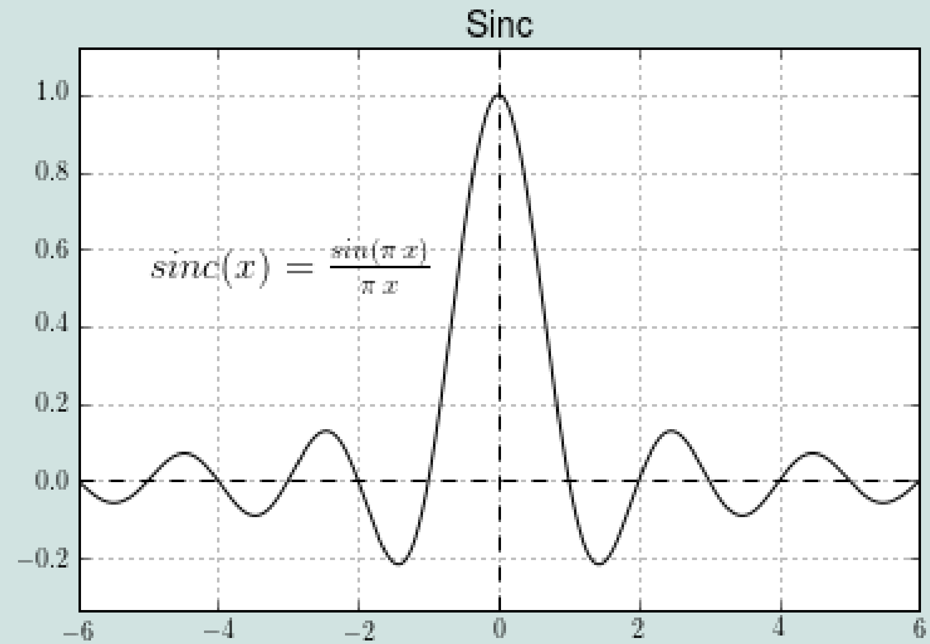
The Fourier transform of a rectangle function

- The Fourier transform of a rectangle function is the sinc function!

$$
\begin{aligned}
\mathscr{F}\{\sqcap\}(s) &= sinc(s) \\
\mathscr{F}^{-1}\{\sqcap\}(x) &= sinc(x) \\
\mathscr{F}\{sinc\}(s) &= \sqcap(s) \\
\mathscr{F}^{-1}\{sinc\}(x) &= \sqcap(x)
\end{aligned}
$$



$\rightleftharpoons$



The Fourier transform of a real-valued function

- The Fourier transform of a real-valued function is a Hermetian function and vice versa

Hermetian means: $f^*(x) = f(-x)$

Real-valued means: $f^*(x) = f(x)$

$$\begin{aligned}
(\mathscr{F}\{f\})^*(s) &= \left(\int_{-\infty}^{+\infty} f(x)\, e^{-\imath 2\pi x s}\, dx\right)^* \\
&= \int_{-\infty}^{+\infty} f^*(x)\, \left[\cos(-2\pi x s) + \imath \sin(-2\pi x s)\right]^*\, dx \\
&= \int_{-\infty}^{+\infty} f^*(x)\, \left[\cos(2\pi x s) - \imath \sin(2\pi x s)\right]^*\, dx \\
&= \int_{-\infty}^{+\infty} f(x)\, \left[\cos(2\pi x s) + \imath \sin(2\pi x s)\right]\, dx \\
&= \int_{-\infty}^{+\infty} f(x)\, \left[\cos(2\pi x(-s)) - \imath \sin(2\pi x(-s))\right]\, dx \\
&= (\mathscr{F}\{f\})(-s)
\end{aligned}$$

The n-dimensional Fourier transform

- The n-dimensional Fourier transformation and its inverse is defined as

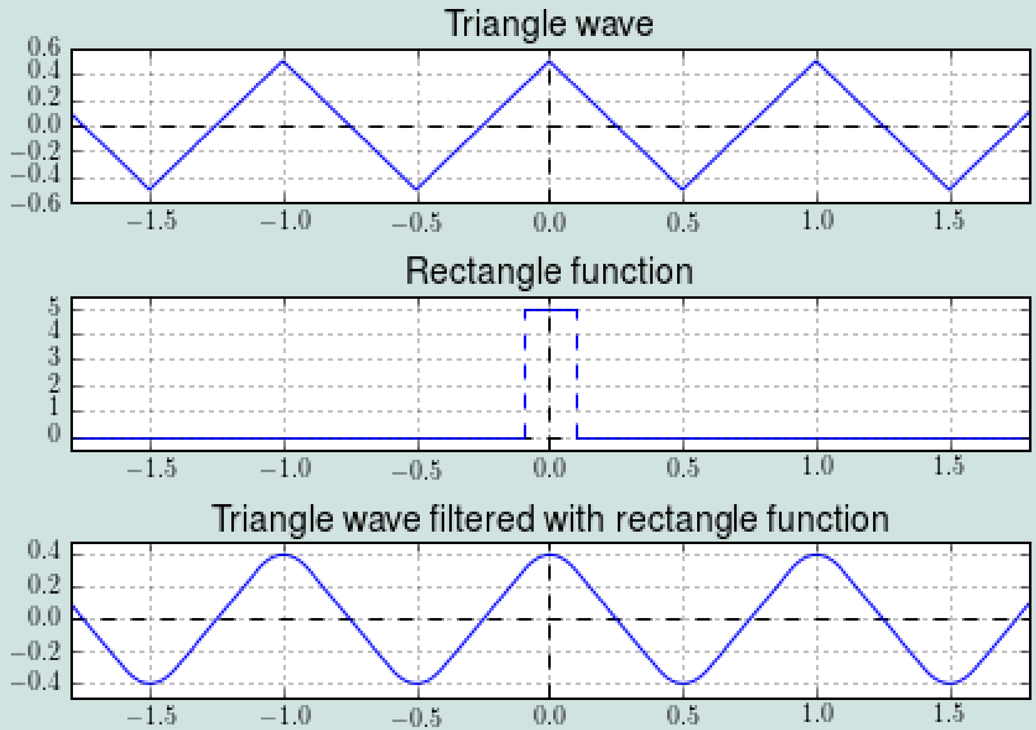
$$\begin{aligned}
\mathscr{F}\{f\}(s_1,\ldots,s_n) &= \mathscr{F}\{f\}(\mathbf{s}) \\
&= \int_{-\infty}^{+\infty} f(x_1,\,\ldots,\,x_n)\,e^{-\imath 2\pi(x_1 s_1+\ldots+x_n s_n)} d^n x \\
&= \int_{-\infty}^{+\infty} f(x)\,e^{-\imath 2\pi(\mathbf{x}\cdot\mathbf{s})} d^n x \\
\mathscr{F}^{-1}\{F\}(x_1,\ldots,x_n) &= \mathscr{F}^{-1}\{F\}(\mathbf{x}) \\
&= \int_{-\infty}^{+\infty} F(s_1,\,\ldots,\,s_n)\,e^{\imath 2\pi(x_1 s_1+\ldots+x_n s_n)} d^n s \\
&= \int_{-\infty}^{+\infty} F(s)\,e^{\imath 2\pi(\mathbf{x}\cdot\mathbf{s})} d^n s
\end{aligned}$$

The Convolution

- The convolution ∘ is the mutual broadening of one function with the other
- Mathematical equivalent of an instrumental broadening or "filtering"

$$\circ : \{f \,|\, f : \mathbb{R} \to \mathbb{C}\} \times \{f \,|\, f : \mathbb{R} \to \mathbb{C}\} \to \{f \,|\, f : \mathbb{R} \to \mathbb{C}\}$$

$$(f \circ g)(x) = \int_{-\infty}^{+\infty} f(x-t)\, g(t)\, dt$$



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$$\circ : \{f \,|\, f : \mathbb{R}^n \to \mathbb{C}\} \times \{f \,|\, f : \mathbb{R}^n \to \mathbb{C}\} \to \{f \,|\, f : \mathbb{R}^n \to \mathbb{C}\} \quad n \in \mathbb{N}$$

$$\begin{aligned}(f \circ g)(x_1, \ldots, x_n) &= (f \circ g)(\mathbf{x}) \\ &= \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} f(x_1 - t_1, \ldots, x_n - t_n)\, g(t_1, \ldots, t_n)\, d^n t \\ &= \int_{-\infty}^{+\infty} f(\mathbf{x} - \mathbf{t})\, g(\mathbf{t})\, d^n t\end{aligned}$$

The Convolution: rules

$$(f \circ g)(x) = \int_{-\infty}^{+\infty} f(x-t)\, g(t)\, dt$$

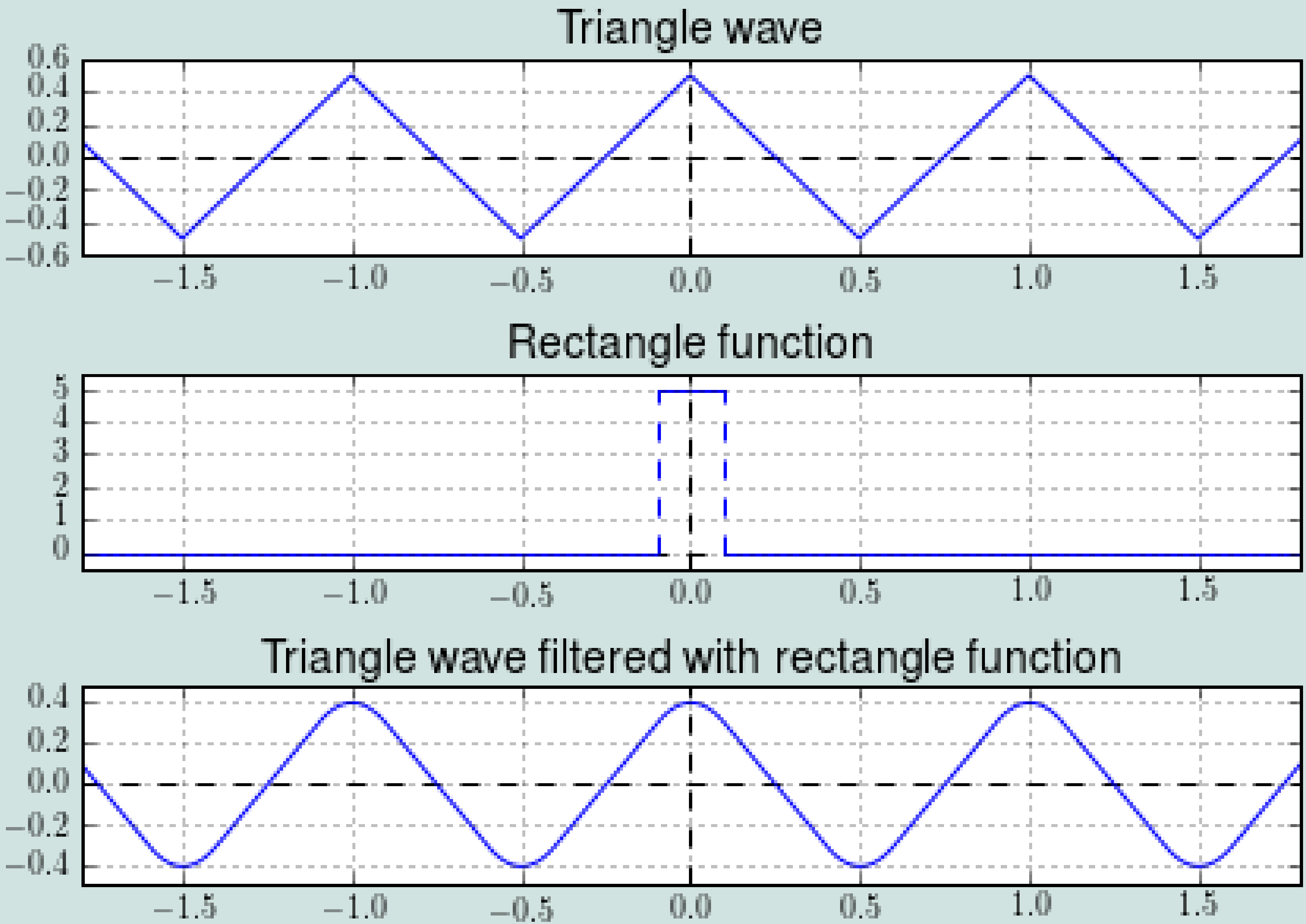
$$f \circ g = g \circ f \qquad \text{(commutativity)}$$

$$(f \circ g) \circ h = f \circ (g \circ h) \qquad \text{(assiociativity)}$$

$$f \circ (g + h) = (f \circ g) + (f \circ h) \qquad \text{(distributivity)}$$

$$(a\, g) \circ h = a\, (g \circ h) \qquad \text{(assiociativity with scalar multiplication)}$$

The Convolution: examples



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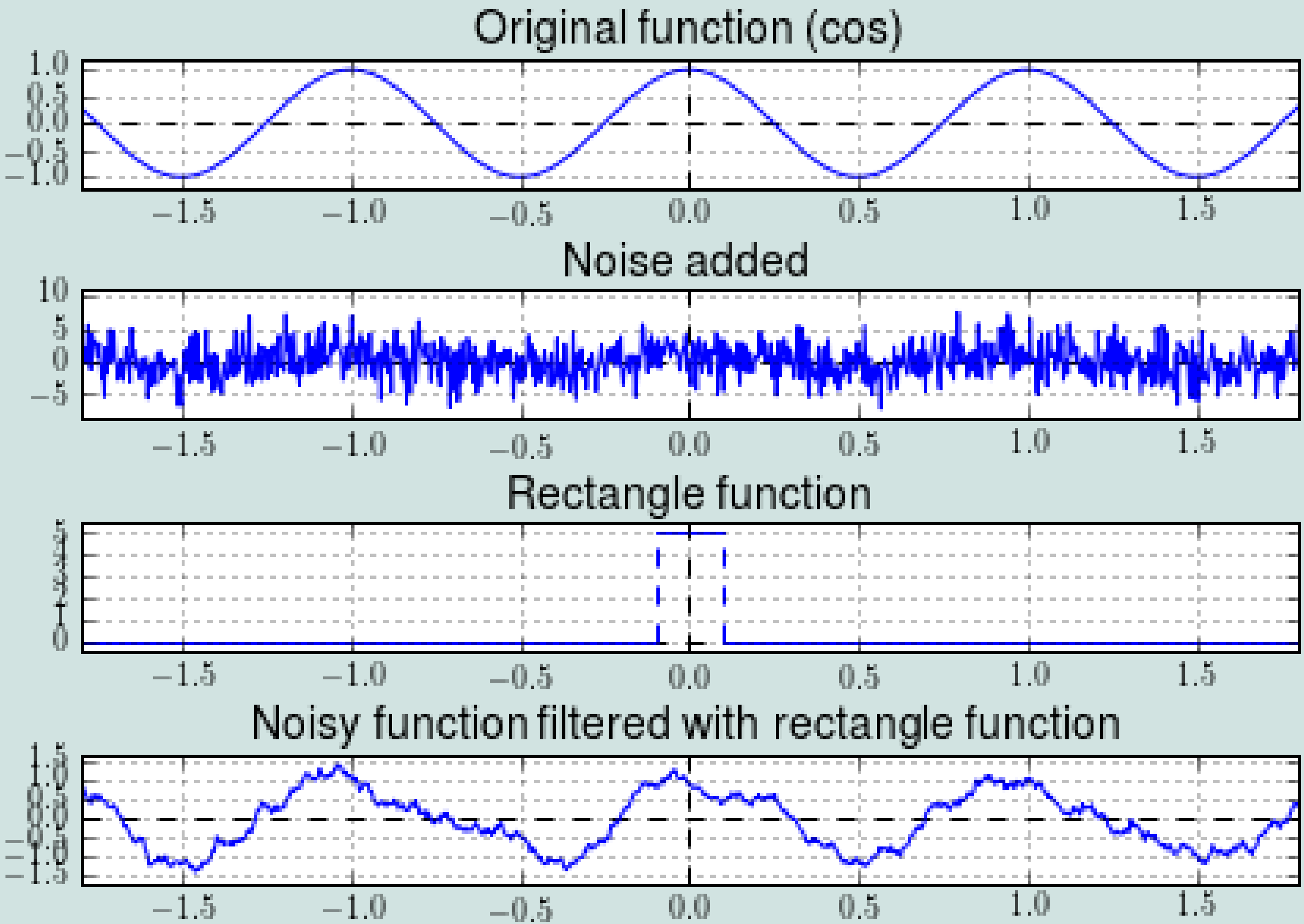
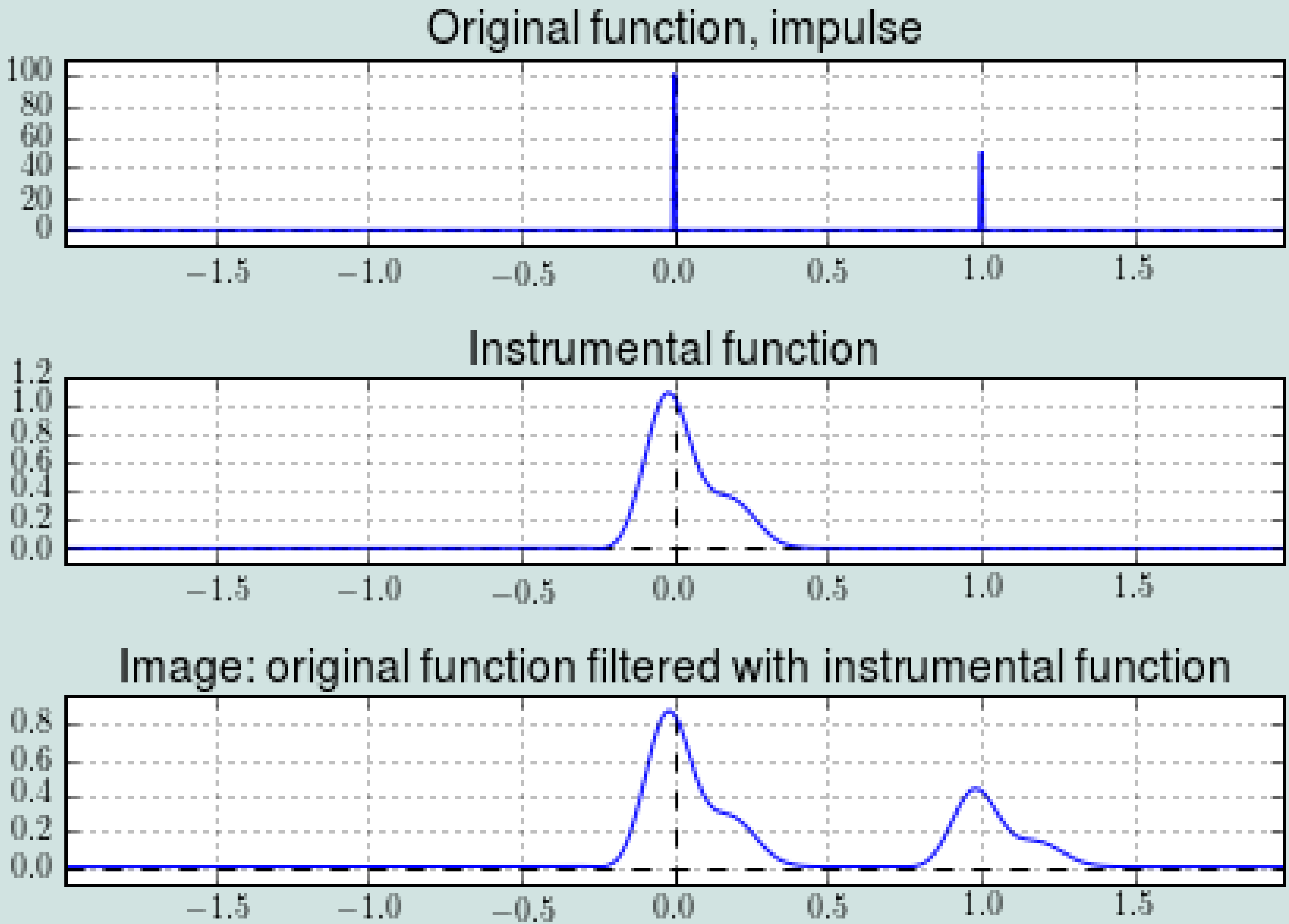


The Convolution: examples



Cross-correlation

$$f_-(x) = f(-x)$$

$$\begin{aligned}(f \star g)(x) &= ({f_-}^* \circ g)(x) \\ &= \int_{-\infty}^{+\infty} f^*(t-x)\, g(t)\, dt \\ &\underset{t'=t-x}{=} \int_{-\infty}^{+\infty} f^*(t')\, g(t'+x)\, dt'\end{aligned}$$

$$\begin{aligned}(f \star g)(x_1, \ldots, x_n) &= (f \star g)(\mathbf{x}) \\ &= ({f_-}^* \circ g)(x) \\ &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f^*(t_1 - x_1, \ldots, t_n - x_n)\, g(t_1, \ldots, t_n)\, d^n t \\ &= \int_{-\infty}^{+\infty} f^*(\mathbf{t} - \mathbf{x})\, g(\mathbf{t})\, d^n t \\ &= \int_{-\infty}^{+\infty} f^*(\mathbf{t})\, g(\mathbf{t} + \mathbf{x})\, d^n t\end{aligned}$$

Cross-correlation

$$f_-(x) = f(-x)$$

$$(f \star g)(x) = (g \star f)^*_-(x)$$

$$\begin{aligned}
(f \star g)(x) &= (f_-{}^* \circ g)(x) \\
&= \int_{-\infty}^{+\infty} f^*(t - x)\, g(t)\, dt \\
&\underset{t'=t-x}{=} \int_{-\infty}^{+\infty} f^*(t')\, g(t' + x)\, dt'
\end{aligned}$$

$$\begin{aligned}
(f \star g)(x_1, \ldots, x_n) &= (f \star g)(\mathbf{x}) \\
&= (f_-{}^* \circ g)(x) \\
&= \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} f^*(t_1 - x_1, \ldots, t_n - x_n)\, g(t_1, \ldots, t_n)\, d^n t \\
&= \int_{-\infty}^{+\infty} f^*(\mathbf{t} - \mathbf{x})\, g(\mathbf{t})\, d^n t \\
&= \int_{-\infty}^{+\infty} f^*(\mathbf{t})\, g(\mathbf{t} + \mathbf{x})\, d^n t
\end{aligned}$$

Auto-correlation

$$f_{-}(x) = f(-x)$$

$$\begin{aligned}(f \star g)(x) &= ({f_{-}}^{*} \circ g)(x) \\ &= \int_{-\infty}^{+\infty} f^{*}(t-x)\, g(t)\, dt \\ &\underset{t'=t-x}{=} \int_{-\infty}^{+\infty} f^{*}(t')\, g(t'+x)\, dt'\end{aligned}$$

$$\begin{aligned}R\{f\}(x) &= (f \star f)(x) \\ &= (f_{-}^{*} \circ f)(x) \\ &= \int_{-\infty}^{+\infty} f^{*}(t-x)\, f(t)\, dt \\ &\underset{t'=t-x}{=} \int_{-\infty}^{+\infty} f^{*}(t')\, f(t'+x)\, dt'\end{aligned}$$

Fourier transform properties: Linearity and separability

$$\mathscr{F}\{f\}(s) = \int_{-\infty}^{+\infty} f(x)\, e^{-\imath 2\pi x s} dx$$

$$\mathscr{F}\{zf\} = z\mathscr{F}\{f\}$$

$$\mathscr{F}\{f+g\} = \mathscr{F}\{f\} + \mathscr{F}\{g\}$$

$$\mathscr{F}\{f\}(s_1, \ldots, s_n) = \mathscr{F}\{f\}(\mathbf{s}) = \int_{-\infty}^{+\infty} f(x_1, \ldots, x_n)\, e^{-\imath 2\pi (x_1 s_1 + \ldots + x_n s_n)} d^n x$$

$$\begin{aligned} f(\mathbf{x}) &= f(x_1, \ldots, x_n) = f_1(x_1) \cdot \ldots \cdot f_n(x_n) \\ &\Rightarrow \\ \mathscr{F}\{f\}(s_1, \ldots, s_n) &= \mathscr{F}\{f_1\}(s_1) \cdot \ldots \cdot \mathscr{F}\{f_n\}(s_n) \end{aligned}$$

Fourier transform properties: Shift theorem

$$
\begin{aligned}
f_{\mathrm{t}}(x) &= f(x-a) \\
\mathscr{F}\{f_{\mathrm{t}}\}(s) &= e^{-2\pi\imath a s}\mathscr{F}\{f\}(s)
\end{aligned}
$$

Proof:

$$
\begin{aligned}
\mathscr{F}\{f_{\mathrm{t}}\}(s) &= \int_{-\infty}^{+\infty} f_{\mathrm{t}}(x)\, e^{-\imath 2\pi x s}\, dx \\
&= \int_{-\infty}^{+\infty} f(x-a)\, e^{-\imath 2\pi x s}\, dx \\
&\underset{x'=x-a}{=} \int_{-\infty}^{+\infty} f(x')\, e^{-\imath 2\pi (x'+a) s}\, \frac{dx}{dx'}\, dx' \\
&= e^{-\imath 2\pi a s} \int_{-\infty}^{+\infty} f(x')\, e^{-\imath 2\pi x' s}\, dx \\
&= e^{-\imath 2\pi a s}\, \mathscr{F}\{f\}(s)
\end{aligned}
$$

Fourier transform properties: Shift theorem

$$f_{\mathrm{t}}(x) = f(x-a)$$
$$\mathscr{F}\{f_{\mathrm{t}}\}(s) = e^{-2\pi\imath as}\mathscr{F}\{f\}(s)$$

$$F_{\mathrm{t}}(s) = F(s-a)$$
$$\mathscr{F}^{-1}\{F_{\mathrm{t}}\}(x) = e^{2\pi\imath ax}\mathscr{F}^{-1}\{F\}(x)$$

Proof:

$$\begin{aligned}
\mathscr{F}\{f_{\mathrm{t}}\}(s) &= \int_{-\infty}^{+\infty} f_{\mathrm{t}}(x)\, e^{-\imath 2\pi xs}\, dx \\
&= \int_{-\infty}^{+\infty} f(x-a)\, e^{-\imath 2\pi xs}\, dx \\
&\underset{x'=x-a}{=} \int_{-\infty}^{+\infty} f(x')\, e^{-\imath 2\pi (x'+a)s}\, \frac{dx}{dx'}\, dx' \\
&= e^{-\imath 2\pi as} \int_{-\infty}^{+\infty} f(x')\, e^{-\imath 2\pi x's}\, dx \\
&= e^{-\imath 2\pi as}\, \mathscr{F}\{f\}(s)
\end{aligned}$$

Fourier transform properties: Convolution theorem

$$\mathscr{F}\{f\}(s) = \int_{-\infty}^{+\infty} f(x)\, e^{-\imath 2\pi x s} dx$$

$$(f \circ g)(x) = \int_{-\infty}^{+\infty} f(x-t)\, g(t)\, dt$$

$$\mathscr{F}\{f \circ g\} = \mathscr{F}\{f\}\mathscr{F}\{g\}$$

Fourier transform properties: Convolution theorem

$$\mathscr{F}\{f\}(s) = \int_{-\infty}^{+\infty} f(x)\, e^{-\imath 2\pi x s} dx$$

$$(f \circ g)(x) = \int_{-\infty}^{+\infty} f(x-t)\, g(t)\, dt$$

$$\mathscr{F}\{f \circ g\} = \mathscr{F}\{f\}\mathscr{F}\{g\}$$

Proof:

$$\begin{aligned}
&\mathscr{F}\{f \circ g\}(s) \\
&= \int_{-\infty}^{+\infty} (f \circ g)(x)\, e^{-\imath 2\pi x s}\, dx \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x-t) g(t)\, dt\, e^{-\imath 2\pi x s}\, dx \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x-t) e^{-\imath 2\pi x s}\, dx\, g(t)\, dt \\
&= \int_{-\infty}^{+\infty} e^{-\imath 2\pi t s}\, \mathscr{F}\{f\}(s)\, g(t)\, dt \\
&= \mathscr{F}\{f\}(s) \int_{-\infty}^{+\infty} g(t)\, e^{-\imath 2\pi t s}\, dt \\
&= \mathscr{F}\{f\}(s) \mathscr{F}\{g\}(s) \\
&= \left(\mathscr{F}\{f\}\mathscr{F}\{g\}\right)(s)
\end{aligned}$$

Fourier transform properties: Convolution theorem

$$\mathscr{F}\{f\}(s) = \int_{-\infty}^{+\infty} f(x)\, e^{-\imath 2\pi x s} dx$$

$$(f \circ g)(x) = \int_{-\infty}^{+\infty} f(x-t)\, g(t)\, dt$$

$$\mathscr{F}\{f \circ g\} = \mathscr{F}\{f\}\mathscr{F}\{g\}$$

$$\mathscr{F}^{-1}\{F \circ G\} = \mathscr{F}^{-1}\{F\}\mathscr{F}^{-1}\{G\}$$

$$\mathscr{F}\{fg\} = \mathscr{F}\{f\} \circ \mathscr{F}\{g\}$$

$$\mathscr{F}^{-1}\{FG\} = \mathscr{F}\{F\} \circ \mathscr{F}\{G\}$$

Proof:

$$\begin{aligned}
&\mathscr{F}\{f \circ g\}(s) \\
&= \int_{-\infty}^{+\infty} (f \circ g)(x)\, e^{-\imath 2\pi x s}\, dx \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x-t) g(t)\, dt\, e^{-\imath 2\pi x s}\, dx \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x-t) e^{-\imath 2\pi x s}\, dx\, g(t)\, dt \\
&= \int_{-\infty}^{+\infty} e^{-\imath 2\pi t s}\, \mathscr{F}\{f\}(s)\, g(t)\, dt \\
&= \mathscr{F}\{f\}(s) \int_{-\infty}^{+\infty} g(t)\, e^{-\imath 2\pi t s}\, dt \\
&= \mathscr{F}\{f\}(s) \mathscr{F}\{g\}(s) \\
&= \left(\mathscr{F}\{f\}\mathscr{F}\{g\}\right)(s)
\end{aligned}$$

Fourier transform properties: Crosscorrelation theorem

$$
\begin{aligned}
(f \star g)(x) &= ({f_-}^* \circ g)(x) \\
&= \int_{-\infty}^{+\infty} f^*(t-x)\, g(t)\, dt \\
&\underset{t'=t-x}{=} \int_{-\infty}^{+\infty} f^*(t')\, g(t'+x)\, dt'
\end{aligned}
$$

$$\mathscr{F}\{f \star g\} = (\mathscr{F}\{f\})^* \cdot \mathscr{F}\{g\}$$

Proof:

$$f_-(x) \underset{\text{def}}{=} f(-x)$$

$$
\begin{aligned}
\mathscr{F}\{f \star g\} &= \mathscr{F}\{{f_-}^* \circ g\} \\
&= \mathscr{F}\{{f_-}^*\} \cdot \mathscr{F}\{g\} \\
&= (\mathscr{F}\{f\})^* \cdot \mathscr{F}\{g\}
\end{aligned}
$$

Fourier transform properties: Autocorrelation theorem

- Also: Wiener-Khinchin Theorem

$$\mathscr{F}\left\{f \star f\right\} = \left|\mathscr{F}\left\{f\right\}\right|^2$$

Proof:

$$\begin{aligned}\mathscr{F}\left\{f \star f\right\} &= \left(\mathscr{F}\left\{f\right\}\right)^* \mathscr{F}\left\{f\right\} \\ &= \left|\mathscr{F}\left\{f\right\}\right|^2\end{aligned}$$

- Just a special case of the cross-correlation theorem

The discrete Fourier transform: definition

- Discrete Fourier transform

$$y = \{y_n \in \mathbb{C}\}_{n=1,\ldots,N}$$
$$\mathscr{F}_{\mathrm{D}}\{y\} = \{Y_k \in \mathbb{C}\}_{k=1,\ldots,N}$$
$$\mathscr{F}_{\mathrm{D}}\{y\}_k = Y_k = \sum_{n=0}^{N-1} y_n \, e^{-\imath 2\pi \frac{nk}{N}}$$

- Inverse discrete Fourier transform

$$Y = \{y_k \in \mathbb{C}\}_{k=1,\ldots,N}$$
$$\mathscr{F}_{\mathrm{D}}^{-1}\{Y\} = \frac{1}{N} \{Y_n \in \mathbb{C}\}_{n \in \mathbb{Z}}$$
$$\mathscr{F}_{\mathrm{D}}^{-1}\{Y\}_n = y_n = \sum_{k=0}^{N-1} Y_k \, e^{\imath 2\pi \frac{nk}{N}}$$

- Numerical methods exist to make the expensive FT faster (“Fast FT”)

The discrete Fourier transform: Inverse

$$
\begin{aligned}
\mathscr{F}_{\mathrm{D}}^{-1}\left\{\mathscr{F}_{\mathrm{D}}\left\{y\right\}\right\}_{n'} &= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{n=0}^{N-1} y_n e^{-\imath 2\pi\frac{kn}{N}}\right) e^{\imath 2\pi \frac{kn'}{N}} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=0}^{N-1}\left(y_n e^{-\imath 2\pi\frac{kn}{N}} e^{\imath 2\pi \frac{kn'}{N}}\right) \\
&= \frac{1}{N}\left(\sum_{k=0}^{N-1} y_{n'} + \sum_{n=0,\, n\neq n'}^{N-1}\sum_{k=0}^{N-1} y_n e^{-\imath 2\pi\frac{kn}{N}} e^{\imath 2\pi \frac{kn'}{N}}\right) \\
&= \frac{1}{N}\left(\sum_{k=0}^{N-1} y_{n'} + \sum_{n=0,\, n\neq n'}^{N-1}\sum_{k=0}^{N-1} y_n e^{\imath 2\pi\frac{k(n'-n)}{N}}\right) \\
&= y_{n'} + \frac{1}{N}\sum_{n=0,\, n\neq n'}^{N-1} y_n \sum_{k=0}^{N-1}\left(e^{\imath 2\pi\frac{(n'-n)}{N}}\right)^k \qquad \left(\sum_{n=0}^{N-1} x^n = \frac{1-x^N}{1-x}\right) \\
&= y_{n'} + \frac{1}{N}\sum_{n=0,\, n\neq n'}^{N-1} y_n \frac{1-\left(e^{\imath 2\pi\frac{(n'-n)}{N}}\right)^N}{1-\left(e^{\imath 2\pi\frac{(n'-n)}{N}}\right)} \\
&= y_{n'} + \frac{1}{N}\sum_{n=0,\, n\neq n'}^{N-1} y_n \frac{1-e^{\imath 2\pi (n'-n)}}{1-e^{\imath 2\pi\frac{(n'-n)}{N}}} \\
&\underset{n,n'\in\mathbb{N}}{=} y_{n'},
\end{aligned}
$$

The discrete Fourier transform and the Fourier transform

- Sample a function with the sampling function s in N regularly spaced steps over the interval

$$\left[x_0 - \frac{N\Delta x}{2}, x_0 + \frac{N\Delta x}{2}\right]$$

$$\begin{aligned}
s_{x_0,\Delta x,N}(x) &= \frac{1}{\Delta x} III\left(\frac{x-x_0}{\Delta x}\right) \cdot \sqcap\left(\frac{x - x_0 + \frac{(N-1)\Delta x}{2}}{(N-1)\Delta x}\right)\\
&= \sum_{n=-\infty}^{+\infty} \delta(x - n\Delta x - x_0) \sqcap\left(\frac{x - x_0 + \frac{(N-1)\Delta x}{2}}{(N-1)\Delta x}\right)\\
&= \sum_{n=0}^{N-1} \delta(x - n\Delta x - x_0)\\
&\Rightarrow\\
f_{\mathrm{s}}(x) &= f \cdot s_{x_0,\,\Delta x,\,N}(x)\\
&= \sum_{n=0}^{N-1} \delta(x - n\Delta x - x_0)\, f(x)
\end{aligned}$$

The discrete Fourier transform and the Fourier transform

$$f_{\mathrm{s}}(x) = \sum_{n=0}^{N-1} \delta(x - n\Delta x - x_0)\, f(x)$$

- Fourier-transform the sampled function

$$\mathscr{F}\left\{f_{\mathrm{s}}\right\}(s) = \sum_{n=0}^{N-1} f(x_0 + n\Delta x)\, e^{-2\pi\imath(x_0 + n\Delta x)s}$$

- Define the set $y := \{y_n \in \mathbb{C}\}_{n=0,\ldots,N-1} = \{f(x_0 + n\Delta x)\}_{n=0,\ldots,N-1}$
- With the discrete FT: $\mathscr{F}_{\mathrm{D}}\{y\}_k = \sum_{n=0}^{N-1} y_n\, e^{-\imath 2\pi \frac{nk}{N}} = \sum_{n=0}^{N-1} f(x_0 + n\Delta x)\, e^{-\imath 2\pi \frac{nk}{N}}$
- We see that if we define $s_k = \dfrac{k}{N\Delta x}$

$$\begin{aligned}
\mathscr{F}\left\{f_{\mathrm{s}}\right\}(s_k) &= \mathscr{F}\left\{f_{\mathrm{s}}\right\}\left(\frac{k}{N\Delta x}\right) \\
&= \mathscr{F}_{\mathrm{D}}\{y\}_k\, e^{-2\pi\imath x_0 s_k} \\
&= \mathscr{F}_{\mathrm{D}}\{y\}_k\, e^{-2\pi\imath \frac{k x_0}{N\Delta x}}
\end{aligned}$$

Nyquist's sampling theorem

- Consider a real-valued wave package with a frequency cutoff at $\frac{\Delta s}{2}$

$$
\begin{aligned}
f(x) &= \int_0^{\frac{\Delta s}{2}} A(s)\,\cos 2\pi\imath s x - \phi(s)\,ds \\
&= \int_0^{\frac{\Delta s}{2}} F(s)\,e^{2\pi\imath s x} + F^*(s)\,e^{-2\pi\imath s x}\,ds \\
&= \int_0^{\frac{\Delta s}{2}} F(s)\,e^{2\pi\imath s x} + F(-s)\,e^{-2\pi\imath s x}\,ds \\
&= \int_{-\frac{\Delta s}{2}}^{\frac{\Delta s}{2}} F(s)\,e^{2\pi\imath s x}\,ds
\end{aligned}
$$

Harry Nyquist

(1889 – 1976)

- The Fourier transform has the support $[-\frac{\Delta s}{2}, \frac{\Delta s}{2}]$ and it follows

$$
\mathscr{F}\{f\}(s) = \mathscr{F}\{f\}(s) \cdot \sqcap(\frac{s}{s_0})
$$

Nyquist's sampling theorem

$$\mathscr{F}\left\{f\right\}(s) = \mathscr{F}\left\{f\right\}(s) \cdot \sqcap(\frac{s}{s_0})$$

- In an experiment we sample the function with the sampling period Δx

$$f_{\mathrm{s}}(x) = f(x) \cdot \frac{1}{\Delta x} III(\frac{x}{\Delta x})$$

$$\begin{aligned}
\mathscr{F}\left\{f_{\mathrm{s}}\right\}(s) &= (\mathscr{F}\left\{f\right\} \circ III_{\Delta x})(s) \\
&= \int_{-\infty}^{\infty} \mathscr{F}\left\{f\right\}(s-t)\, III(\Delta x t)\, dt \\
&= \int_{-\infty}^{\infty} \mathscr{F}\left\{f\right\}(s-t)\, \frac{1}{\Delta x} \left(\sum_{n=-\infty}^{+\infty} \delta\left(t - \frac{n}{t}\right) \right) dt \\
&= \frac{1}{\Delta x} \sum_{-\infty}^{+\infty} \mathscr{F}\left\{f\right\}\left(s - \frac{n}{\Delta x}\right)
\end{aligned}$$

$$(= \frac{1}{\Delta x} \sum_{-\infty}^{+\infty} \mathscr{F}\left\{f\right\}\left(s - \frac{n}{\Delta x}\right) \cdot \sqcap\left(\frac{s - \frac{n}{\Delta x}}{s_0}\right))$$

Nyquist's sampling theorem

- In an experiment we sample the function with the sampling period Δx

$$\mathscr{F}\{f_\mathrm{s}\}(s) = \frac{1}{\Delta x}\sum_{-\infty}^{+\infty}\mathscr{F}\{f\}\left(s-\frac{n}{\Delta x}\right)$$

$$(= \frac{1}{\Delta x}\sum_{-\infty}^{+\infty}\mathscr{F}\{f\}\left(s-\frac{n}{\Delta x}\right)\cdot\sqcap\left(\frac{s-\frac{n}{\Delta x}}{s_0}\right))$$

- The Fourier transform repeats itself, it is aliased.

- If we sample a function with the bandwidth $\frac{\Delta s}{2}$, the sampling interval has to fulfil the condition

$$\frac{1}{\Delta x} > \Delta s$$

Limited sampling

- The thought experiment is not yet realistic. We can only measure for a limited number of samples

$$f_{\mathrm{sc}} = f_{\mathrm{s}} \cdot \sqcap\left(\frac{x - \frac{(N-1)\Delta x}{2}}{(N-1)\Delta x}\right)$$

$$= \sum_{n=0}^{N-1} \delta\left(x - n\Delta x\right) f(x)$$

- It follows:

$$\mathscr{F}\left\{f_{\mathrm{sc}}\right\}(s) = \mathscr{F}\left\{f_{\mathrm{s}}\right\} \circ \left((N-1)\Delta x\, sinc\left(\left(N-1\right)\Delta x s\right) e^{\mp 2\pi \frac{(N-1)\Delta x}{2}}\right)$$

- The Fourier transform is hence always filtered with a sinc function, which gets narrower with increasing number of samples

Literature

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